

DECOMPOSITION OF QUANTUM GATES INTO CONTROLLED QUBIT GATES

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Abstract

It is shown that every d -by- d unitary matrix can be written as the product of two-level unitary matrices with additional structure and prescribed determinants. The result is then applied to show that every quantum (unitary) gate acting on n -qubit register can be expressed as no more than $2^n(2^n - 1)/2$ controlled single qubit gates chosen from $2^n - 1$ classes so that the quantum gates in each class share the same $n - 1$ control qubits. Related results are discussed.

Keywords Quantum gates, controlled qubit gates, two-level unitary matrices, Gray codes.

1 Introduction

The foundation of quantum computation [9] involves the encoding of computational tasks into the temporal evolution of a quantum system. A register of qubits, identical two-state quantum systems, is employed, and quantum algorithms can be described by unitary transformations and projective measurements acting on the 2^n -dimensional state vector of the register. In this context, unitary transformations (matrices) are also called quantum gates.

It is well known [1] that the set of single qubit gates and CNOT gate are universal. In other words, any unitary gate acting on an n -qubit register can be implemented with single qubit gates and CNOT gates. Typically, the proof can be done in the following steps; for example, see [8, §4.6] and [9, pp.188-193].

Step 1. Every $d \times d$ (special) unitary matrix can be written as a product of no more than $d(d - 1)/2$ two-level (special) unitary matrices. (A two-level unitary matrix is obtained from I_d by changing a two-by-two principal submatrix.)

Step 2. Every quantum gate of size 2^n corresponding to a two-level unitary matrix U can be converted to a controlled qubit gate \tilde{U} using at most $n - 1$ swaps of states by the Gray code technique. (Here each swap is a product of single qubit gates and CNOT gates.) Then the controlled qubit gate \tilde{U} can be expressed as the product of single qubit gates and CNOT gates. One can then reverse the swaps of states to get the decomposition of the unitary U into single qubit gates and CNOT gates.

There has been considerable interest in finding efficient and practical algorithms in constructing unitary gates; see [2, 3, 5, 6, 7, 8, 9, 12] and the references therein. The purpose of this note is to show that one can express any unitary gate (matrix) acting on n -qubit register as no more than $2^n(2^n - 1)/2$ controlled single qubit gates. These gates can be chosen from $2^n - 1$ classes such that each of these classes consists of controlled qubit gates using the same $n - 1$ control qubits.

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While our proposed construction does not improve much in terms of the complexity of the decomposition of unitary gates (because controlled qubit gates are still expensive to implement), it gives a more elegant proof for the universality of (selected) controlled qubit gates. Moreover, our proof uses only elementary properties of unitary matrices, and a clever combinatorial idea to decompose a given unitary gate as the product of (selected) controlled qubit gates. The method is easy to understand and implement. For example, a Matlab program for decomposing a 4-by-4 unitary matrix into no more than 6 controlled qubit gates by Ms. Rebecca Roberts is available at <http://cklixx.people.wm.edu/mathlib.html>.

To make our method more portable to other problems, we present a general lemma which allows one to decompose a unitary matrix to two-level unitary matrices with additional structure and prescribed determinants. The technique and insight are useful in other problems in quantum information science and related subjects.

We will present our results, proofs, and examples in Section 2, and give a short discussion in Section 3.

2 Results, Examples, and Proofs

Let $P = (j_1, j_2, \dots, j_d)$ be such that the entries of P correspond to a permutation of $(1, 2, \dots, d)$. A two-level unitary matrix is called a P -unitary matrix of type k for $k \in \{1, 2, \dots, d-1\}$ if it is obtained from I_d by changing a principal submatrix with row and column indexes j_k and j_{k+1} . For example, if $P = (j_1, j_2, j_3, j_4) = (1, 2, 4, 3)$, then the three types of P -unitary matrices of type 1, 2, and 3 have the forms

$$\begin{pmatrix} * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & 0 & * \\ 0 & 0 & 1 & 0 \\ 0 & * & 0 & * \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix},$$

respectively. We have the following.

Lemma 2.1 *Let $P = (j_1, j_2, \dots, j_d)$ be such that the entries of P correspond to a permutation of $(1, 2, \dots, d)$. Then every d -by- d unitary matrix U can be written as a product of no more than $d(d-1)/2$ P -unitary matrices. Moreover, those P -unitary matrices can be chosen to have any determinants with modulus 1 as long as their product equals $\det(U)$.*

An immediate consequence of the lemma is the following.

Corollary 2.2 *Every $d \times d$ special unitary matrix can be written as a product of no more than $d(d-1)/2$ P -unitary matrices each of which has determinant 1.*

It is instructive to illustrate a special case of the lemma. We consider the case when $d = 4$ and $P = (j_1, j_2, j_3, j_4) = (1, 2, 4, 3)$ as above. (As we will see, this example is relevant to our later discussion on quantum gates.)

Let

$$U = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{pmatrix}$$

be a four-by-four unitary matrix. Let $\mu_1, \mu_2, \dots, \mu_6$ be such that $\mu_1, \mu_2, \dots, \mu_6 \in \{z : |z| = 1\}$ and $\mu_1 \mu_2 \cdots \mu_6 = \det(U)$. We divide the construction into two steps.

Step 1. We consider the column of U labeled by the first entry of P (i.e., the first column).

Choose P -unitary matrix U_1 of type 3 with $\det(U_1) = \bar{\mu}_1$ such that the $(j_4, j_1) = (3, 1)$ entry of $U_1 U$ is 0 as follows. Let $u_1 = \sqrt{|a_{31}|^2 + |a_{41}|^2}$ and

$$U_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{\bar{\mu}_1 a_{41}}{u_1} & \frac{-\bar{\mu}_1 a_{31}}{u_1} \\ 0 & 0 & \frac{a_{31}}{u_1} & \frac{a_{41}}{u_1} \end{pmatrix}. \quad \text{Then} \quad U_1 U = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ u_1 & a'_{42} & a'_{43} & a'_{44} \end{pmatrix}.$$

Next choose P -unitary matrix U_2 of type 2 such that $(j_3, j_1) = (4, 1)$ entry of $U_2 U_1 U$ is 0 as follows. Let $u_2 = \sqrt{|a_{21}|^2 + u_1^2}$, and

$$U_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{\bar{a}_{21}}{u_2} & 0 & \frac{u_1}{u_2} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{-\bar{\mu}_2 u_1}{u_2} & 0 & \frac{\bar{\mu}_2 a_{21}}{u_2} \end{pmatrix}. \quad \text{Then} \quad U_2 U_1 U = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ u_2 & a'_{22} & a'_{23} & a'_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a''_{42} & a''_{43} & a''_{44} \end{pmatrix}.$$

Now, choose P -unitary matrix U_3 of type 1 so that the $(j_2, j_1) = (2, 1)$ entry of $U_3 U_2 U_1 U$ is also 0, and the $(1, 1)$ entry of $U_3 U_2 U_1 U$ equals 1 as follows. Let

$$U_3 = \begin{pmatrix} \bar{a}_{11} & u_2 & 0 & 0 \\ -\bar{\mu}_3 u_2 & \bar{\mu}_3 a_{11} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \quad \text{Then} \quad V = U_3 U_2 U_1 U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a''_{22} & a''_{23} & a''_{24} \\ 0 & a'_{32} & a'_{33} & a'_{34} \\ 0 & a''_{42} & a''_{43} & a''_{44} \end{pmatrix}.$$

Note that the first row of V has the form $(1, 0, 0, 0)$ because V is unitary.

Step 2. We turn to columns of V labeled by $j_2 = 2$ and $j_3 = 4$.

Choose P -unitary matrices U_4, U_5 of types 3 and 2 with determinants $\bar{\mu}_4$ and $\bar{\mu}_5$, respectively, so that the $(j_4, j_2) = (3, 2)$ entry of $U_4 V$ and the $(j_3, j_2) = (4, 2)$ entry of $U_5 U_4 V$ are 0. Then choose a P -unitary matrix U_6 of type 3 with determinant $\bar{\mu}_6$ so that the $(j_4, j_3) = (3, 4)$ entry of $U_6 U_5 U_4 V$ is 0. Here are the zero patterns of the matrices in the process:

$$U_4 V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & * & * & * \end{pmatrix}, \quad U_5 U_4 V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix}, \quad U_6 U_5 U_4 V = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Note that in the last step, the $(j_4, j_4) = (3, 3)$ entry of $U_6 U_5 U_4 V$ is 1 because

$$\det(U_6 U_5 U_4 U_3 U_2 U_1) \det(U) = \bar{\mu}_6 \bar{\mu}_5 \cdots \bar{\mu}_1 \det(U) = |\det(U)|^2 = 1.$$

Consequently,

$$U = U_1^\dagger U_2^\dagger U_3^\dagger U_4^\dagger U_5^\dagger U_6^\dagger.$$

Clearly, each U_j^\dagger is a P -unitary matrix of the same type as $U_j, j = 1, 2, \dots, 6$.

Obviously, we can skip some of the P -unitary matrices if the entry to be eliminated is already 0 during the process. We can now present the proof of Lemma 2.1.

Proof of Lemma 2.1. Let $P = (j_1, j_2, \dots, j_d)$ where the entries of P is a permutation of $(1, 2, \dots, d)$. Let U be a d -by- d unitary matrix. We extend the construction in the example to the general case as follows.

Step 1. First consider the j_1 th column of U . One can choose P -unitary matrices $U_{d-1}, U_{d-2}, \dots, U_1$ of types $d-1, d-2, \dots, 1$ with prescribed determinants to eliminate the entries of U in positions $(j_d, j_1), (j_{d-1}, j_1), \dots, (j_2, j_1)$ successively, so that the j_1 th column of the matrix $V = U_1 U_2 \dots U_{d-1} U$ equals the j_1 th column of I_d . Since V is unitary, the j_1 th row of V will equal the j_1 th row of I_d .

Step 2. Consider the j_2 th column of V . One can choose P -unitary matrices $V_{d-1}, V_{d-2}, \dots, V_2$ of types $d-1, d-2, \dots, 2$ with prescribed determinants to eliminate the entries of V in positions $(j_d, j_2), (j_{d-1}, j_2), \dots, (j_3, j_2)$ successively, so that the j_2 th column of the matrix $W = V_2 \dots V_{d-1} V$ equals the j_2 th column of I_d . Since W is unitary, the j_2 th row of W will equal the j_2 th row of I_d .

We can repeat this process in converting the j_3 th \dots, j_{d-1} th columns to the j_3 th \dots, j_{d-1} th column of I_d successively, by P -unitary matrices. Finally, the (j_d, j_d) element will also be 1 by the determinant condition imposed on the P -unitary matrices. Multiplying the inverses of the P -unitary matrices in the appropriate orders, we see that U is a product of P -unitary matrices as asserted, and the number of P -unitary matrices used is no more than $(d-1) + \dots + 1 = d(d-1)/2$ because some of the P -unitary matrices may be chosen to be identity if the entry to be eliminated is already 0 during the process. \square

We will apply Lemma 2.1 to the decomposition of a quantum gate into the product of controlled qubit gates. In this application, we need a special selection of the vector $P = (i_1, i_2, \dots, i_d)$.

In the following discussion, we always assume that n is a positive integer and $N = 2^n$. Recall that a Gray code G_n [10] is defined as an N -tuple $G_n = (X_1, X_2, \dots, X_N)$ such that

- (a) each X_1, X_2, \dots, X_N are length n binary sequences corresponding to binary representation of the numbers $0, 1, \dots, N-1$, arranged in a certain order,
- (b) two adjacent sequences X_j and X_{j+1} differ in only one position for each $j = 1, 2, \dots, N-1$, and
- (c) the sequences X_N and X_1 differ in only one position.

One can construct a Gray code G_n recursively as follows.

Set $G_1 = (0, 1)$; for $n \geq 1$ and $N = 2^n$, if $G_n = (X_1, \dots, X_{N-1}, X_N)$, set

$$G_{n+1} = (0X_1, \dots, 0X_{N-1}, 0X_N, 1X_N, 1X_{N-1}, \dots, 1X_1).$$

For example, we have

$$G_2 = (00, 01, 11, 10), \quad G_3 = (000, 001, 011, 010, 110, 111, 101, 100), \text{ etc.}$$

Let us adapt the definition of P -unitary matrices to define G_n -unitary matrices, which correspond to controlled single qubit gates in quantum information science. To this end, label the rows and columns of an $N \times N$ matrix by the binary numbers

$$0 \dots 0, 0 \dots 01, \dots, 1 \dots 1.$$

An $N \times N$ two-level unitary matrix is a G_n -unitary matrix of type k if it differs from I_N by a principal submatrix with rows and columns labeled by two consecutive terms X_k and X_{k+1} in the Gray code $G_n = (X_1, X_2, \dots, X_N)$, $k \in \{1, 2, \dots, N-1\}$. Clearly, there are $N-1$ types of G_n -unitary matrices. Since X_k and X_{k+1} differ in only one position, every G_n -unitary matrix corresponds to a controlled single qubit gate.

For example, G_2 -unitary matrices of types 1, 2, 3 have the following forms, respectively:

$$\begin{array}{ccc} \begin{array}{c} 00 \ 01 \ 10 \ 11 \\ 00 \begin{pmatrix} * & * & 0 & 0 \\ 01 \begin{pmatrix} * & * & 0 & 0 \\ 10 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 11 \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \end{array} \end{array}, & \begin{array}{c} 00 \ 01 \ 10 \ 11 \\ 00 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 01 \begin{pmatrix} 0 & * & 0 & * \\ 10 \begin{pmatrix} 0 & 0 & 1 & 0 \\ 11 \begin{pmatrix} 0 & * & 0 & * \end{pmatrix} \end{array} \end{array}, & \begin{array}{c} 00 \ 01 \ 10 \ 11 \\ 00 \begin{pmatrix} 1 & 0 & 0 & 0 \\ 01 \begin{pmatrix} 0 & 1 & 0 & 0 \\ 10 \begin{pmatrix} 0 & 0 & * & * \\ 11 \begin{pmatrix} 0 & 0 & * & * \end{pmatrix} \end{array} \end{array} \end{array} \end{array}.$$

Note that these are exactly the P -unitary matrices when $P = (1, 2, 4, 3)$.

It is now easy to adapt Lemma 2.1 to prove the following.

Theorem 2.3 *Let n be a positive integer and $N = 2^n$. Every $N \times N$ unitary matrix U is a product of m G_n -unitary matrices with $m \leq N(N-1)/2$. Furthermore, the G_n -unitary matrices can be chosen to have any determinant with modulus 1 as long as their product equals $\det(U)$.*

Proof. Identify $(0, 1, \dots, N-1)$ with the N -tuple of binary numbers $(0 \cdots 0, 0 \cdots 01, \dots, 1 \cdots 1)$; label the rows and columns of U by the binary numbers $0 \cdots 0, 0 \cdots 01, \dots, 1 \cdots 1$. Then apply Lemma 2.1 to U with P replaced by G_n . \square

Since every G_n -unitary matrix corresponds to a controlled single qubit gate, by Theorem 2.3 every unitary gate can be written as product of controlled single qubit gates from $2^n - 1$ classes so that matrices in each class share the same $n-1$ control qubits.

The example before the proof of Lemma 2.1 can be viewed as an illustration of Theorem 2.3 when $n = 2$ if we identify $P = (1, 2, 4, 3)$ and $G_2 = (00, 01, 11, 10)$. An illustration of our construction for $n = 3$ is given in the Appendix.

3 Discussion

In this note, we obtained a decomposition of a d -by- d unitary matrix as product of special structures specified by a vector $P = (j_1, j_2, \dots, j_d)$ such that the entries of P correspond to a permutation of $(1, 2, \dots, d)$. The result is then applied to show that every unitary gate U acting on n qubits can be decomposed as product of special two-level unitary matrices corresponding to controlled single qubit gates. This was done by using Gray code $G_n = (X_1, X_2, \dots, X_N)$ with $N = 2^n$, and constructing G_n -unitary matrices which are two-level matrices obtained from I_N by changing its principal submatrix with row and column indexes X_k and X_{k+1} for $k = 1, 2, \dots, N-1$.

There are other applications of Lemma 2.1. For example, if $P = (1, 2, \dots, d)$, then P -unitary matrices are two-level tridiagonal unitary matrices. In numerical linear algebra and other applications, it is useful to decompose a matrix into tridiagonal forms with simple structure; e.g., see [4, 11] and their references. In our study, the focus is on quantum information science, and the decomposition of the unitary matrix respects the tensor (Kronecker) product structure of matrices. The same technique may be useful for decomposition of matrices with other multilinear structures.

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Appendix: Illustration of Theorem 2.3 for three qubit unitary gates

Let us illustrate Theorem 2.3 when $n = 3$ in the following. It will further demonstrate how to eliminate the nonzero entries of the given unitary matrix using the G_n vector, which will determine the order of the columns to be treated, and the order of the entries to be eliminated in a selected column. Evidently, depending on the applications, one may use different Gray codes for the decomposition of a given unitary gate or a special class of unitary gates.

To illustrate Theorem 2.3 when $n = 3$, note that

$$G_3 = (X_1, X_2, \dots, X_8) = (000, 001, 011, 010, 110, 111, 101, 100).$$

Suppose U is an eight-by-eight unitary matrix. We divide the decomposition of U in several steps.

Step 1. First, we consider column $X_1 = 000$ (i.e., the first column). Choose G_3 -unitary matrices $U_{X_1}^{X_7}, U_{X_1}^{X_6}, U_{X_1}^{X_5}, U_{X_1}^{X_4}, U_{X_1}^{X_3}, U_{X_1}^{X_2}, U_{X_1}^{X_1}$, where the upper index in each symbol corresponds to the type of the G_3 -unitary matrix, with appropriate determinants (as done in the case when $n = 2$) to eliminate the nonzero entries in column X_1 with row indexes $X_8, X_7, X_6, X_5, X_4, X_3, X_2$, successively. Here note that if we identify the row and column indexes $(000, 001, \dots, 111)$ with $(1, 2, \dots, 8)$, then the Gray code $(X_1, X_2, X_3, X_4, X_5, X_6, X_7, X_8)$ corresponds to $(1, 2, 4, 3, 7, 8, 6, 5)$. So, we are eliminating the entries in the first column lying in rows 5, 6, 8, 7, 3, 4, 2, successively. Let $U_{X_1} = U_{X_1}^{X_1} U_{X_1}^{X_2} U_{X_1}^{X_3} U_{X_1}^{X_4} U_{X_1}^{X_5} U_{X_1}^{X_6} U_{X_1}^{X_7}$. Then

$$U_{X_1}U = \begin{array}{cccccccc} 000 & 001 & \dots & \dots & \dots & \dots & 111 \\ \left(\begin{array}{cccccccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * & * \end{array} \right) & \begin{array}{l} 000 \\ 001 \\ 010 \\ 011 \\ 100 \\ 101 \\ 110 \\ 111 \end{array} \end{array}.$$

Step 2. We consider column $X_2 = 001$ (i.e., the second column). Choose G_3 -unitary matrices $U_{X_2}^{X_7}, U_{X_2}^{X_6}, U_{X_2}^{X_5}, U_{X_2}^{X_4}, U_{X_2}^{X_3}, U_{X_2}^{X_2}$ with appropriate determinants to eliminate the nonzero entries in column X_2 with row indexes $X_8, X_7, X_6, X_5, X_4, X_3$, successively. (Here, using the usual row and column indexes of an eight-by-eight matrix, we have eliminated the nonzero entries in the second column with row indexes 5, 6, 8, 7, 3, 4 successively.) Let $U_{X_2} = U_{X_2}^{X_2} U_{X_2}^{X_3} U_{X_2}^{X_4} U_{X_2}^{X_5} U_{X_2}^{X_6} U_{X_2}^{X_7}$. Then

$$U_{X_2}U_{X_1}U = \begin{pmatrix} I_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \\ 0 & * & * & * & * & * & * \end{pmatrix}.$$

Step 3. We consider column $X_3 = 011$ (i.e., the fourth column). Choose G_3 -unitary matrices $U_{X_3}^{X_7}, U_{X_3}^{X_6}, U_{X_3}^{X_5}, U_{X_3}^{X_4}, U_{X_3}^{X_3}$ with appropriate determinants to eliminate the nonzero entries in column X_3

with row indexes X_8, X_7, X_6, X_5, X_4 . Let $U_{X_3} = U_{X_3}^{X_3} U_{X_3}^{X_4} U_{X_3}^{X_5} U_{X_3}^{X_6} U_{X_3}^{X_7}$. Then

$$U_{X_3} U_{X_2} U_{X_1} U = \begin{pmatrix} I_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * & * & * \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & * & 0 & * & * & * & * \\ 0 & * & 0 & * & * & * & * \\ 0 & * & 0 & * & * & * & * \\ 0 & * & 0 & * & * & * & * \end{pmatrix}.$$

Step 4. We consider column $X_4 = 010$ (i.e., the third column). Choose G_3 -unitary matrices $U_{X_4}^{X_7}$, $U_{X_4}^{X_6}$, $U_{X_4}^{X_5}$, $U_{X_4}^{X_4}$ with appropriate determinants to eliminate the nonzero entries in column X_4 with row indexes X_8, X_7, X_6, X_5 . Let $U_{X_4} = U_{X_4}^{X_4} U_{X_4}^{X_5} U_{X_4}^{X_6} U_{X_4}^{X_7}$. Then

$$U_{X_4} U_{X_3} U_{X_2} U_{X_1} U = \begin{pmatrix} I_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \\ 0 & 0 & 0 & * & * & * & * \end{pmatrix}.$$

Step 5. We consider column $X_5 = 110$ (i.e., the seventh column). Choose G_3 -unitary matrices $U_{X_5}^{X_7}$, $U_{X_5}^{X_6}$, $U_{X_5}^{X_5}$ with appropriate determinants to eliminate the nonzero entries in column X_5 with row indexes X_8, X_7, X_6 . Let $U_{X_5} = U_{X_5}^{X_5} U_{X_5}^{X_6} U_{X_5}^{X_7}$. Then

$$U_{X_5} U_{X_4} U_{X_3} U_{X_2} U_{X_1} U = \begin{pmatrix} I_4 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * \\ 0 & * & * & 0 & * \\ 0 & 0 & 0 & 1 & 0 \\ 0 & * & * & 0 & * \end{pmatrix}.$$

Step 6. We consider column $X_6 = 111$ (i.e., the eighth column). Choose G_3 -unitary $U_{X_6}^{X_7}$, $U_{X_6}^{X_6}$ with appropriate determinants to eliminate the nonzero entries in column X_6 with row indexes X_8, X_7 . Let $U_{X_6} = U_{X_6}^{X_6} U_{X_6}^{X_7}$. Then

$$U_{X_6} U_{X_5} U_{X_4} U_{X_3} U_{X_2} U_{X_1} U = \begin{pmatrix} I_4 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & * & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Step 7. Finally, we consider column $X_7 = 101$ (i.e., the sixth column). We choose a G_3 -unitary U_{X_7} with appropriate determinant to eliminate the nonzero entry in the (X_8, X_7) position so that

$$U_{X_7} U_{X_6} U_{X_5} U_{X_4} U_{X_3} U_{X_2} U_{X_1} U = \begin{pmatrix} I_4 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Consequently, we have $U = U_{X_1}^\dagger U_{X_2}^\dagger \cdots U_{X_7}^\dagger$, and the construction is completed.